5th Australasian Congress on Applied Mechanics, ACAM 2007
10-12 December 2007, Brisbane, Australia

Timoshenko beam-solution in terms of integrated radial basis functions

A. Khennane[1] **and T. Tran-Cong**[2]

[1]Aerospace Civil and Mechanical Engineering, UNSW@ADFA, Northcott Drive, Canberra
[2]Faculty of Engineering and Surveying, University of Southern Queensland, Toowoomba

Abstract: The Indirect Radial Basis Function (IRBF) method is a mesh free method that operates on the strong form of a differential equation. It uses a point interpolation method, which makes the implementation of boundary conditions very easy. It does not require any numerical integration as the method starts with the interpolation of the highest order derivatives of the unknown function with radial basis functions. The lesser order derivatives and the function itself are obtained using symbolic integration, which ensures no loss of accuracy. The method is used to solve the Timoshenko beam. Typical results for uniformly distributed loads are compared to similar predictions obtained from analytical solutions. Excellent agreement is obtained for a variety of boundary conditions.

Keywords: Radial basis functions, mesh free methods, IRBF, Timoshenko beam

1 Introduction

Radial basis functions (RBF) have been widely used for the interpolation of functions and the solution of differential equations [1-6] to cite only a few references. Recently RBF are also being used for the solution of structural mechanics problems [7-9]. As part of the rapidly evolving field of meshless methods, they do present a real advantage in the use of a grid of arbitrary points to represent the domain rather than the use of a mesh of elements with predefined connectivity.

Nearly all RBF applications in structural mechanics use the direct approach. That is; the unknown function is approximated with RBF and the derivatives obtained by subsequent differentiations of the trial function. However, as shown by Mai Duy and Tran-Cong [3,4] the direct approach is less accurate than the indirect approach (IRBF) for the solution of differential equations. The indirect approach (IRBF) uses radial basis functions to approximate the highest order derivatives. The lesser order derivatives and the function itself are obtained using symbolic integrations. However, in addition to the unknown weights, this method also results in unknown constants of integration. This work focus on the application of the indirect approach (IRBF) to the solution of the differential equation of bending: Timoshenko beam.

2 Integrated basis radial functions

Given a function of many variables, $\mathbf{F}(\mathbf{x}) = \mathbf{F}(x^{(1)}, x^{(2)}, ..., x^{(p)})$. The first derivative of the function with respect to the variable $x^{(k)}$ can be interpolated using radial basis functions as:

$$\mathbf{F}_{,k}(\mathbf{x}) = \frac{\partial \mathbf{F}(\mathbf{x})}{\partial x^{(k)}} = \sum_{j=1}^{m} w_j g_j(\mathbf{x}) \tag{1}$$

where $\left\{w_j\right\}_{j=1}^{m}$ represent a set of weights and $g_j(\mathbf{x})$ is a set of radial basis functions which can be written in a general form as $g_j(\mathbf{x}) = \mathrm{f}_j\left(\left\|\mathbf{x} - \mathbf{c}_j\right\|\right)$ where $\|.\|$ denotes the Euclidean norm and $\left\{\mathbf{c}_j\right\}_{j=1}^{m}$ is a set of the centres that can be chosen from among the known data points $\left\{x_i \in \mathbf{R}^p \ (i=1, 2, ..., n\right\}$. When Multiquadrics are used, the radial basis function $g_j(\mathbf{x})$ is written as :

$$g_j(\mathbf{x}) = \mathrm{f}_j\left(\left\|\mathbf{x}-\mathbf{c}_j\right\|\right) = \sqrt{r^2 + a_j^2} \quad \text{for some} \quad a_j > 0 \tag{2}$$

The original function can then be obtained as:

$$\mathbf{F}(\mathbf{x}) = \int \frac{\partial \mathbf{F}(\mathbf{x})}{\partial x^{(k)}} dx^{(k)} = \int \sum_{j=1}^{m} w_j g_j(\mathbf{x}) dx^{(k)} = \sum_{j=1}^{m} w_j \mathbf{H}_j(\mathbf{x}) + C_1(x^{(1)}, \ldots . x^{(k-1)}, x^{(k+1)}, \ldots ., x^{(p)}) \tag{3}$$

The function $\mathbf{H}_j(\mathbf{x})$ is obtained by symbolically integrating the function $g_j(\mathbf{x})$.

Integrating four times the function $g_j(\mathbf{x})$ with respect to the variable $x^{(k)}$ yields:

$$\mathbf{H}_j(\mathbf{x}) = \int g_j(\mathbf{x}) dx^{(k)} + C_1 \tag{9}$$

$$\bar{\mathbf{H}}_j(\mathbf{x}) = \int \mathbf{H}_j(\mathbf{x}) dx_k + C_1 x^{(k)} + C_2 \tag{10}$$

$$\bar{\bar{\mathbf{H}}}_j(\mathbf{x}) = \int \bar{\mathbf{H}}_j(\mathbf{x}) dx_k + \frac{C_1}{2} x^{(k)2} + C_2 x^{(k)} + C_3 \tag{11}$$

$$\bar{\bar{\bar{\mathbf{H}}}}_j(\mathbf{x}) = \int \bar{\bar{\mathbf{H}}}_j(\mathbf{x}) dx_k + \frac{C_1}{6} x^{(k)3} + \frac{C_2}{2} x^{(k)2} + C_3 x^{(k)} + C_4 \tag{12}$$

The symbolic integrations are carried out using Mathematica®.[10]. The integration constants are functions of all the variables except $x^{(k)}$.

3 Application of the indirect approach method to the Timoshenko beam

3.1 Governing equations

The Timoshenko beam model includes a first order correction for transverse shear. The key cinematic assumption is that plane sections remain plane but no longer normal to the neutral axis. As a result, the total rotation is made up of the gradient of the deflection and the shear rotation; that is:

$$\mathrm{q} = \frac{dy}{dx} + \mathrm{g} \tag{13}$$

The stress displacement relations are given by the following:

$$M = EI \frac{d\mathrm{q}}{dx} \tag{14}$$

$$S = GA_S \mathrm{g} = GA_S \left(\mathrm{q} - \frac{dy}{dx}\right) \tag{15}$$

where G is the shear modulus, A_S the effective shear area.

For equilibrium, the following relations must hold:

$$\frac{dM}{dx} = S = EI \frac{d^2\mathrm{q}}{dx^2} = GA_S \left(\mathrm{q} - \frac{dy}{dx}\right) \tag{16}$$

$$\frac{dS}{dx} = EI \frac{d^3\mathrm{q}}{dx^3} = p(x) \tag{17}$$

3.2 Boundary conditions

The boundary conditions for the Timoshenko beam are given as follow:

Free end (F): $M = S = 0$ (18)

Simply supported end (S): $$y = M = \frac{d\theta}{dx} = 0 \tag{19}$$

Clamped end (C): $$y = \frac{dy}{dx} = \theta = 0 \tag{20}$$

3.3 Numerical implementation

The third derivatives in θ is interpolated using radial basis functions. The second and first derivatives and the function θ itself are obtained using symbolic integration. That is for $\theta(x)$:

$$\frac{d^3\theta}{dx^3}(x) = \sum_{j=1}^{m} w_j^{\theta} g_j(x) \tag{21}$$

$$\frac{d^2\theta}{dx^2}(x) = \int \sum_{j=1}^{m} w_j^{\theta} g_j(x)dx = \sum_{j=1}^{m} w_j^{\theta} \int g_j(x)dx + C_1 = \sum_{j=1}^{m} w_j^{\theta} H_j(x) + C_1 \tag{22}$$

$$\frac{d\theta(x)}{dx} = \sum_{j=1}^{m} w_j^{\theta} \int H_j(x)dx + \int C_1 dx = \sum_{j=1}^{m} w_j^{\theta} \overline{H}_j(x) + C_1 x + C_2 \tag{23}$$

$$\theta(x) = \sum_{j=1}^{m} w_j^{\theta} \int \overline{H}_j(x)dx + \int C_1 x dx + \int C_2 dx = \sum_{j=1}^{m} w_j^{\theta} \overline{\overline{H}}_j(x) + C_1 \frac{x^2}{2} + C_2 x + C_3 \tag{24}$$

Introducing the dimensionless factor $\Omega = \frac{EI}{GA_S}$, the deflection y can be obtained from (16) as:

$$y(x) = \int \left(\theta - \Omega \frac{d^2\theta}{dx^2} \right) dx + C_4 \tag{25}$$

Using (22) and (24), the deflection is interpolated as:

$$\begin{aligned} y(x) &= \int \left(\sum_{j=1}^{m} w_j^{\theta} \overline{\overline{H}}_j(x) + C_1 \frac{x^2}{2} + C_2 x + C_3 \right) dx - \Omega \int \left(\sum_{j=1}^{m} w_j^{\theta} H_j(x) + C_1 \right) dx \\ &= \left(\sum_{j=1}^{m} w_j^{\theta} \overline{\overline{\overline{H}}}_j(x) + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4 \right) - \Omega \left(\sum_{j=1}^{m} w_j^{\theta} \overline{H}_j(x) + C_1 x + C_2 \right) \end{aligned} \tag{26}$$

Discretising equation (17) using equation (21) yields:

$$EI \left(\sum_{j=1}^{m} w_j^{\theta} g_j(x_i) \right) = p(x_i) \tag{27}$$

Forcing the boundary conditions at $x = 0$, and $x = L$, adds in four equations, which are given for a simply supported end as:

At $x = 0$

$$y_{(x=0)} = \sum_{j=1}^{m} w_j^{\theta} \overline{\overline{\overline{H}}}_j(x=0) - \Omega \sum_{j=1}^{m} w_j^{\theta} \overline{H}_j(x=0) + C_1 \left(\frac{0^3}{6} - \Omega 0 \right) + C_2 \left(\frac{0^2}{2} - \Omega \right) + C_3 0 + C_4 = 0 \tag{28}$$

$$\frac{d\theta}{dx}_{(x=0)} = \sum_{j=1}^{m} w_j^{\theta} \overline{H}_j(0) + C_1 0 + C_2 = 0 \tag{29}$$

At $x = L$

$$y_{(x=L)} = \sum_{j=1}^{m} w_j^{\theta} \overline{\overline{\overline{H}}}_j(x=L) - \Omega \sum_{j=1}^{m} w_j^{\theta} \overline{H}_j(x=L) + C_1 \left(\frac{L^3}{6} - \Omega L \right) + C_2 \left(\frac{L^2}{2} - \Omega \right) + C_3 L + C_4 = 0 \tag{30}$$

$$\frac{dq}{dx}_{(x=L)} = \sum_{j=1}^{m} w_j^{q} \bar{H}_j(L) + C_1 L + C_2 = 0 \tag{31}$$

For a simply supported beam (SS) discretised with n points $x_i \in [0,L]$ and two boundary points $x = 0$ and $x = L$, combining equations (27) to (31) gives the complete system of equations in terms of the unknown weights w_j and the constants C_1 to C_4.

$$\begin{bmatrix} \bar{\bar{\bar{H}}}_1(x=0) - \text{W}\bar{H}_1(x=0), & \bar{\bar{\bar{H}}}_2(x=0) - \text{W}\bar{H}_2(x=0), & —, & —, & \bar{\bar{\bar{H}}}_m(x=0) - \text{W}\bar{H}_m(x=0), & 0, & 0, & 0, & 1 \\ \bar{\bar{\bar{H}}}_1(x=L) - \text{W}\bar{H}_1(x=L), & \bar{\bar{\bar{H}}}_2(x=L) - \text{W}\bar{H}_2(x=L), & —, & —, & \bar{\bar{\bar{H}}}_m(x=L) - \text{W}\bar{H}_m(x=L), & L^3/6 - \text{W}L, & L^2/2 - \text{W}, & L, & 1 \\ \bar{H}_1(x=0), & \bar{H}_2(x=0), & —, & —, & \bar{H}_m(x=0), & 0, & 1, & 0, & 0 \\ \bar{H}_1(x=L), & \bar{H}_2(x=L), & —, & —, & \bar{H}_m(x=L), & L, & 1, & 0, & 0 \\ g_1(x_1), & g_2(x_1), & —, & —, & g_m(x_1), & 0, & 0, & 0, & 0 \\ g_1(x_2), & g_2(x_2), & —, & —, & g_m(x_2), & 0, & 0, & 0, & 0 \\ ——, & ——, & —, & —, & ——, & 0, & 0, & 0, & 0 \\ g_1(x_{n-2}), & g_2(x_{n-2}), & —, & —, & g_m(x_{n-2}), & 0, & 0, & 0, & 0 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ — \\ — \\ w_m \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{p(x_1)}{EI} \\ — \\ — \\ -\frac{p(x_{n-2})}{EI} \end{Bmatrix} \tag{32}$$

The system consists of $n+4$ equations and $m+4$ unknowns, and for $n = m$, the matrix is square and the system of equations can be easily solved.

4 Numerical results

To validate the derived numerical models, the uniformly loaded deep beam shown on Figure 1 is analysed with different boundary conditions: (SS) simply supported at both ends, (CS) clamped at one end, simply supported at the other, (CC) clamped at both ends, and (CF) clamped at one, free at the other.

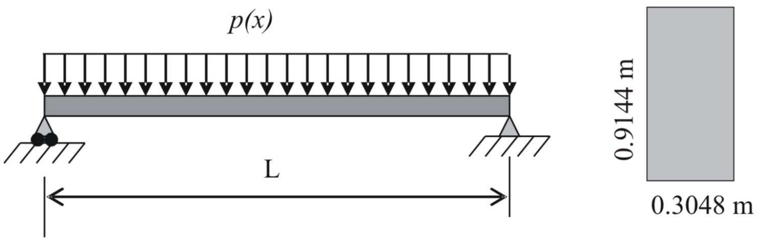

Figure 1: Simply supported beam under uniform loading

The geometrical domain of the beam is discretised with 20 interpolation points including the two boundary nodes. For the sake of numerical computations, a material having an elastic modulus $E = 40.e{+}6\ kN/m^2$ and a Poisson's ratio $\text{n} = 0.2$ was chosen. The results obtained with the models with their analytical counterparts are shown respectively on Figures 2 and 3 in the form of diagrams representing the deflection and the rotation for the (SS) and (CC) beams. The exact solutions are obtained from [11,12].

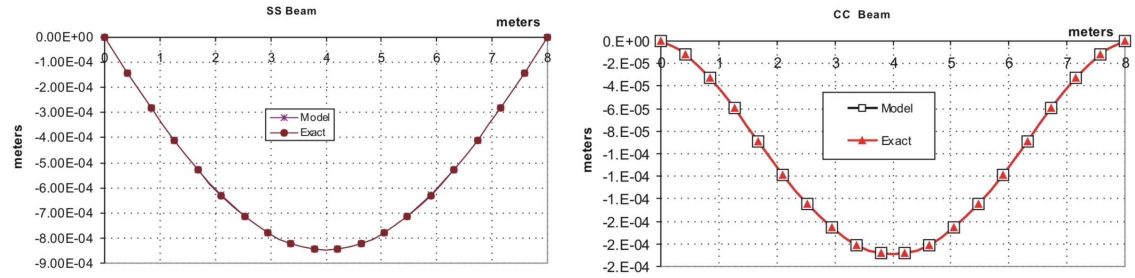

Figure 2: Predicted deflections versus exact solutions for the (SS) and (CC) beams

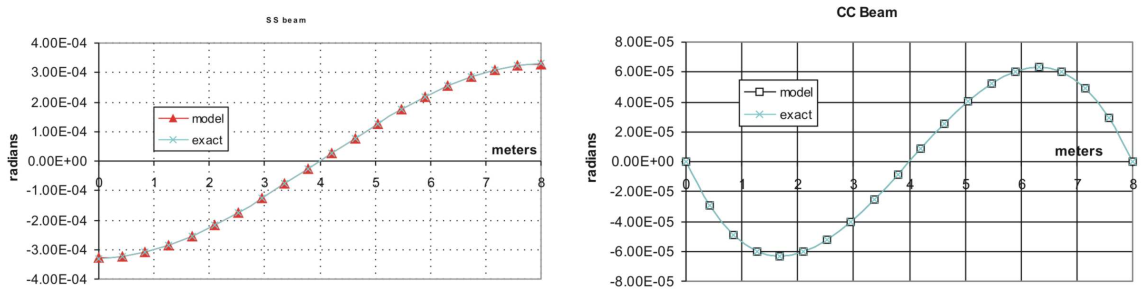

Figure 3: Predicted rotations versus exact solutions for the (SS) and (CC) beams

When compared to the analytical solutions, it can be seen that the indirect approach is extremely precise. The accuracy is such that the two solutions (exact and model) are not discernable on the graphs. As an indication of the degree of precision, a measure of the error of the solution is computed in the form of equation (33) for the simply supported (SS) beam for two different discretisations; respectively 20 and 100 interpolation points.

$$e = \sqrt{\sum_{i}\left(U_{E}(x_{i}) - U_{M}(x_{i})\right)^{2} / \sum_{i}\left(U_{E}(x_{i})\right)^{2}} \qquad (33)$$

The results are shown on Table -1.

		Timoshenko model
deflection	Error (n= 20 interpolation points) Error (n= 100 interpolation points)	1.31E-3 7.69E-6
rotation	Error (n= 20 interpolation points) Error (n= 100 interpolation points)	1.39e-3 8.94e-6

Table 1: Relative errors between the computed and analytical results.

With only 20 interpolation points, the error is very small and is in the order 1.3e-3. Increasing the number of interpolation points to 100 results in an insignificant error of the order of 7.6e-6 for the deflection and 8.9e-6 for the rotation. This high degree of accuracy results from the fact that the indirect method interpolates the higher order derivative while the lower order derivatives and the function itself are obtained with symbolic integration. Indeed, it is well known that it is much easier to approximate a function rather than its derivative. But when the derivative is interpolated, the function is much easier to approximate. In addition, the use of symbolic integration reduces round-off errors resulting in a greater accuracy. In this aspect the IRBF method has proved to be very powerful.

5 Conclusion

The Indirect Radial Basis Function (IRBF) method is applied to the differential equations of bending: Timoshenko model. The model involves derivatives of the unknown function up to the third order. The IRBF method uses radial basis functions to interpolate the highest order derivative while the lesser order derivative and the function itself are obtained using symbolic integrations. Symbolic integration eliminates the need to resort to numerical integration, henceforth reducing round-off errors and resulting in greater accuracy. The method is also very easy to use. As input, it only requires a set of RBF centres and collocation points, which can be evenly or randomly distributed. Furthermore, it uses a point interpolation method, which makes the implementation of boundary conditions very easy.

References

[1] Kansa, E.J. (1990) “Multiquadrics – a scattered data approximation scheme with application to computational fluid dynamics, i: surface approximations and partial derivative estimates. Comput. Math. Appl. 19(8/9) pp:127-145

[2] Kansa, E.J. (1990) “Multiquadrics – a scattered data approximation scheme with application to computational fluid dynamics, i: solution to parabolic, hyperbolic and elliptic partial differential equations. Comput. Math. Appl. 19(8/9) pp:147-161

[3] Mai-Duy N, Tran-Cong T. (2001). "Numerical solution of differential equations using multiquadric radial basis function networks". *Neural Networks*; **14**:185–199.

[4] Mai-Duy N, Tran-Cong T. (2001) “Numerical solution of Navier-Stokes equations using radial basis function networks”. *Int. J. Numer. Fluids*; **37**:65–86.

[5] Mai-Duy N, Tran-Cong T. (2003).”Approximation of function and its derivatives using radial basis function networks”. *Appl. Math. Modelling*; **27**(3):197–220.

[6] Mai-Duy N, Tran-Cong T. (2003). "Indirect RBFN method with thin plate spline for numerical solution of differential equations". *Computer Modelling in Engineering and Science* **4**(1):85–102.

[7] Leitão, V.M.A. (2001) “A meshless method for Kirchhoff plate bending problems”. Int. J. Numer. Meth. Engng. 52 pp:1107-1130

[8] Ferreira, A.J.M. (2003) “A formulation of the multiquadric radial basis function method for the analysis of laminated composite plates”. Composite Structures, 59: 385-392

[9] Liew, K.M., Chen X.L. and Reedy, J.N. (2004). “ Mesh free radial basis function method for buckling analysis of non-uniformly loaded arbitrarily shaped shear deformable plates”. Computer Methods in Applied Mechanics and Engineering. Vol. 193, No 3-5, pp:205-224

[10] Wolfram S. *Mathematica Version 4.* Wolfram Research. http://www.wolfram.com

[11] Timoshenko, S.P and Goodier J.N. (1970) “Theory of elasticity”. Third Edition, McGraw Hill International Editions, Singapore

[12] Wang C.M. (1995) “Timoshenko beam bending solutions in terms of Euler Bernoulli solutions” ASCE Journal of Engineering Mechanics Vol. 121 No 6, pp:763-765